

CHAPTER 1

Single-RF Longitudinal Dynamics

In this chapter, we provide an introduction to longitudinal dynamics in particle accelerators. Slip-stacking is a complex problem in longitudinal dynamics that is best understood as a perturbation on the single-RF system that is presented in this chapter.

1.1 RF Accelerating Cavities

Particle beams are accelerated in high-energy machines by devices known as radiofrequency (RF) accelerating cavities. RF cavities are resonant devices used to trap an oscillating electromagnetic field which acts on the charged particle beam via the Lorentz force. RF cavities trap electromagnetic waves only at discrete frequencies (eigenfrequencies), with corresponding spatiotemporal distributions (eigenmodes). Typically, RF cavities are only operated at the lowest order mode, known as the fundamental or TM_{010} mode, with higher order modes being unwanted deviations from simple longitudinal acceleration. We will use the term RF frequency $\omega_{rf} = 2\pi f_{rf}$ to refer specifically to the accelerating mode.

Fig. 1.1 shows a simple RF cavity operating in the TM_{010} mode. The electric field is parallel (or antiparallel) to the motion of the beam and the magnetic field is perpendicular and rotationally symmetric. In a purely cylindrical cavity, the RF frequency is determined by the geometry of the cylinder. However the ferrite tuner is electromagnetic coupled to the body of the cavity and it can be used to adjust the RF frequency of the cavity. A

bias current is applied through the ferrite tuner which adjusts the magnetic permeability of the ferrite rings and consequently shifts the RF frequency as needed. Other RF cavities change the resonant frequency with mechanical tuners. The electromagnetic waves generate heat due to the surface resistance of the RF cavity and this heat is compensated by cooling elements that are not shown in Fig. 1.1.

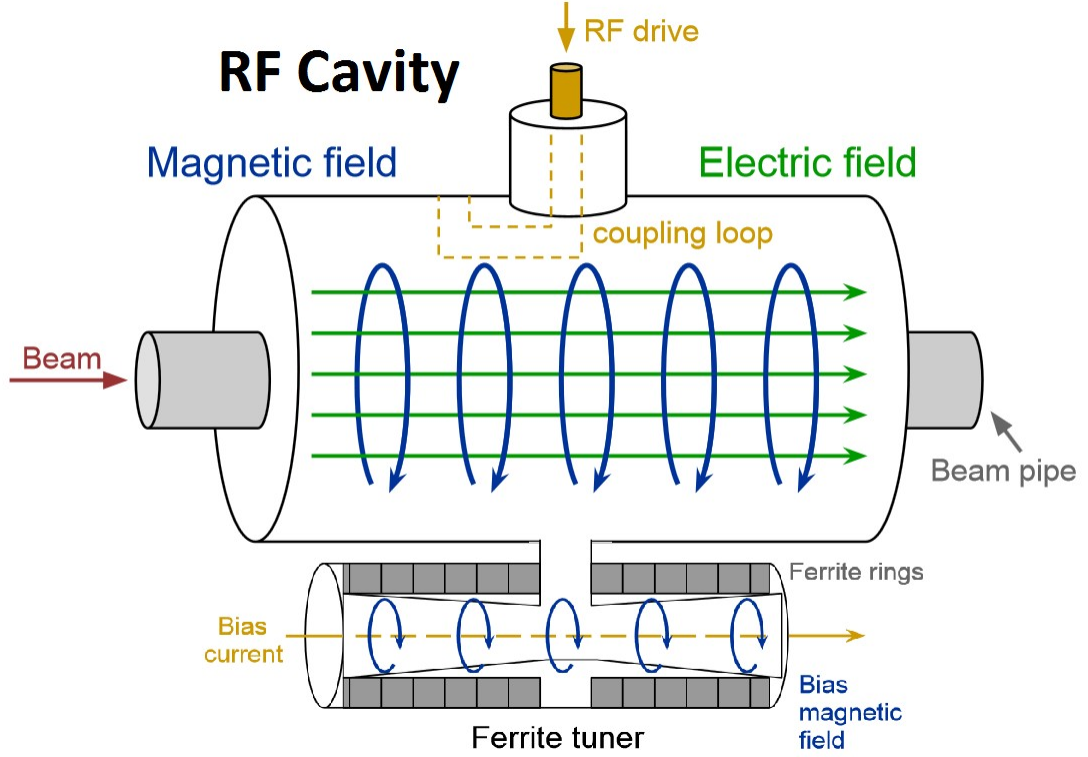


Figure 1.1: A diagram of a simplified RF cavity. [1]

High-voltage RF electronics are used to amplify a sinusoidal signal and drive the RF cavity. From an electronic perspective, the RF cavity can be approximated by the RLC resonant circuit shown in Fig. 1.2. The corresponding resonance frequency is given by

$$f_{rf} = \frac{1}{2\pi\sqrt{L_{eq}C_{eq}}} \quad (1.1)$$

where L_{eq} is equivalent inductance and C_{eq} is the equivalent capacitance. Another important parameter is the quality factor of the cavity, known as Q , which is the ratio of energy stored

to energy dissipated per cycle. From the circuit model, the Q -factor is given by:

$$Q = R_{sh} \frac{C_{eq}}{L_{eq}} \quad (1.2)$$

where R_{sh} is the shunt impedance. As shown in Fig. 1.2, a higher Q -factor corresponds to a sharper the frequency resonance.

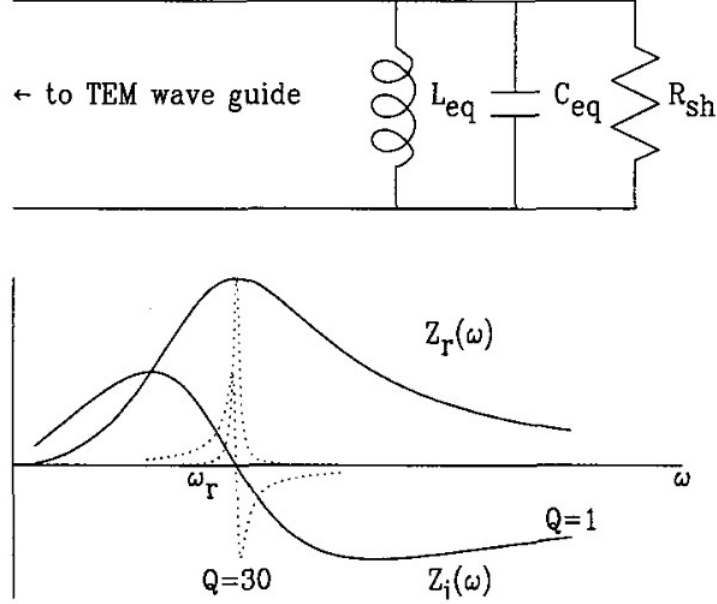


Figure 1.2: (top) An equivalent RLC circuit corresponding to a resonant RF cavity. (bottom) The complex impedance of the RLC circuit for $Q = 1$ and $Q = 30$. [2]

1.2 Derivation of Longitudinal Motion

Here we derive the longitudinal motion of charged particles accelerated by an RF cavity. For now, we consider only the single-particle motion and then we relate our results to the ensemble of particles that make up a charged particle beam. The longitudinal coordinates of a charged particle are a position coordinate ϕ and a momentum coordinate δ . The ϕ coordinate is the arrival time of a charged particle relative to the oscillation of RF wave in the cavity. The δ coordinate is the fractional deviation of the particle momentum from a

reference momentum.

The acceleration of a charged particle passing through a time-varying accelerating field is given by

$$\Delta E = qV \frac{\beta c}{d} \int_{-d/2\beta c}^{d/2\beta c} \sin(\omega_{rf}t + \phi) dt \quad (1.3)$$

where q is the charge of particle, V is the RF cavity voltage, βc is the velocity of the particle, d is the length of the accelerating gap in the RF cavity, and ϕ is the phase corresponding to the arrival time of the incoming particle. The integral given in Eq. 1.3 is trivial to solve:

$$\Delta E = qV(\sin \chi/\chi) \sin(\phi) \quad (1.4)$$

where $\chi = \omega_{rf}d/2\beta c$. The term $\sin \chi/\chi$ is referred to as the transit time factor and hereafter we use the effective cavity voltage $V(\sin \chi/\chi) \rightarrow V$.

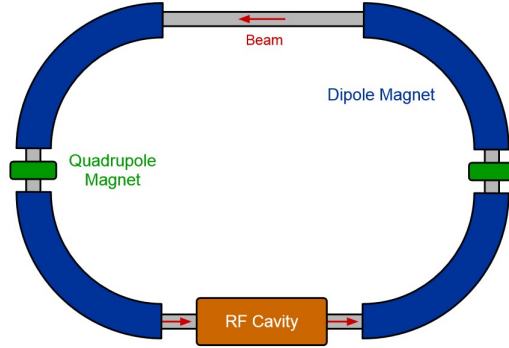


Figure 1.3: A simplified particle accelerator ring. The dipole magnets curve the beam so that it can be accelerated many times by the same RF cavity. The quadrupole magnets transversely focus the beam so it does not diverge into the beampipe wall as it is transported. [1]

In a cyclic particle accelerator a particle beam is directed along a closed curve and passes through the same sequence of beamline elements each revolution. Fig. 1.3 shows a diagram of a simple ring, where dipoles are used to bend the beam in a loop, quadrupoles are used to focus the beam transversely, and an RF cavity is used to accelerate the beam. In order

for an RF cavity to act coherently on a particle over many passes, the RF frequency f_{rf} of the particle must be near a harmonic multiple h of the revolution frequency f_{rev} . When $f_{rf} = hf_{rev}$ we refer to this as the reference frequency and the corresponding momentum of the particle as the synchronous momentum p_0 . We define our momentum coordinate δ as the fractional deviation from the synchronous momentum:

$$\delta \equiv \frac{p - p_0}{p_0} \quad (1.5)$$

Using Eq. 1.4 and Eq. 1.5 we write the change in the delta coordinate per unit time.

$$\begin{aligned} \dot{\delta} &= \frac{\dot{p}}{p_0} = \frac{\dot{E}}{\beta^2 E_0} = f_{rev} \frac{\Delta E}{\beta^2 E_0} = f_{rev} \frac{qV}{\beta^2 E_0} \sin(\phi) \\ \dot{\delta} &= f_{rev} V_\delta \sin(\phi) \end{aligned} \quad (1.6)$$

where $V_\delta = \frac{qV}{\beta^2 E_0}$ is the maximum fractional change in the reference momentum within a single revolution.

The revolution period depends on the particle momentum and determines the change in the phase ϕ during each revolution. The phase-slip factor η is defined to be the linear dependence of the revolution period on particle momentum:

$$\frac{T - T_{rev}}{T_{rev}} \approx 0 + \frac{1}{T_{rev}} \frac{\partial T}{\partial \delta} \delta = \eta \delta \quad (1.7)$$

Recall that we've defined the reference momentum such that $T = T_{rev}$ when $\delta = 0$. Particles at greater momentum take less time to travel the same path length but generally take longer path lengths through particle accelerators. Consequently, the phase-slip factor η can be positive or negative but will generally increase with energy:

$$\eta = \frac{1}{T_{rev}} \frac{\partial T}{\partial \delta} = \frac{1}{C} \frac{\partial C}{\partial \delta} - \frac{1}{\beta} \frac{\partial \beta}{\partial \delta} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} \quad (1.8)$$

where C is the path length of the particle beam in one revolution and is referred to as the circumference. The parameter γ_T is calculated from the properties of the accelerator lattice

and corresponds to the energy at which the phase-slip factor is zero. A synchrotron is a cyclic particle accelerator that is designed to operate with a nonzero phase-slip factor. For a non-accelerating beam we take η to be a given constant parameter.

Using Eq. 1.7 and taking η to be constant, we can write the change in the ϕ -coordinate as a function of the δ -coordinate:

$$\begin{aligned}\dot{\phi} &= f_{rev}\Delta\phi = 2\pi f_{rev}\frac{\Delta T}{T_{rf}} = 2\pi f_{rev}h\frac{\Delta T}{T_{rev}} \\ \dot{\phi} &= 2\pi f_{rev}h\eta\delta\end{aligned}\tag{1.9}$$

1.3 Longitudinal Equations of Motion

Taking Eq. 1.6 and Eq. 1.9 together, we find the complete equations of motion in the form of coupled first-order differential equations:

$$\dot{\delta} = f_{rev}V_{\delta}\sin(\phi), \quad \dot{\phi} = 2\pi f_{rev}h\eta\delta\tag{1.10}$$

The corresponding second-order equation of motion is

$$\ddot{\phi} = -\omega_s^2\sin(\phi)\tag{1.11}$$

where

$$\omega_s = 2\pi f_{rev}\sqrt{\frac{V_{\delta}h|\eta|}{2\pi}}\tag{1.12}$$

The frequency of small oscillations ω_s is referred to as the synchrotron frequency. We have adopted the convention that if $\eta > 0$ then $\phi \rightarrow \phi + \pi$ so that the stable fixed point is always obtained at $\phi = 0$ and the unstable fixed point is always found at $\phi = \pi$. Clearly Eq. 1.11 describes a system isomorphic to the simple pendulum. Fig. 1.4 shows the trajectories of particles governed by Eq. 1.10.

For small ϕ , Eq. 1.11 has a stable solution known as a synchrotron oscillation

$$\phi = \rho\sin(\omega_s t + \psi)\tag{1.13}$$

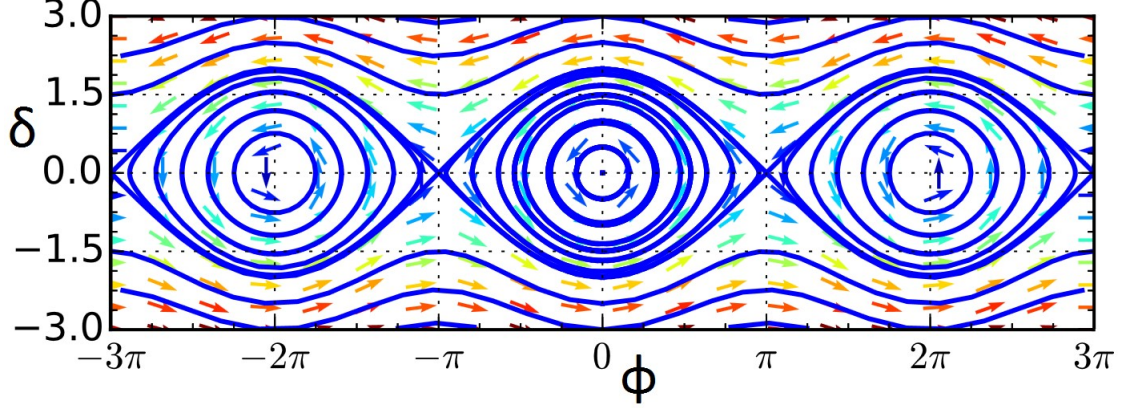


Figure 1.4: Several phase-space trajectories for the synchrotron motion (blue lines) alongside the instantaneous phase-space motion (arrows). Here, $\eta < 0$.

where the amplitude ρ and the initial phase ψ are set by initial conditions.

The Hamiltonian corresponding to Eq. 1.10 is given by

$$H = \pi f_{rev} h |\eta| \delta^2 + f_{rev} V_\delta [1 - \cos(\phi)] \quad (1.14)$$

The sign of Eq. 1.14 has been fixed (for positive and negative η) so that H is nonnegative and is zero only at the stable fixed point. The separatrix ($H = 2f_{rev}V_\delta$) is given by

$$\delta = \pm \sqrt{\frac{V_\delta}{\pi h |\eta|}} \sqrt{1 + \cos(\phi)} = \pm \frac{2}{h |\eta|} \frac{\omega_s}{\omega_{rev}} \cos\left(\frac{\phi}{2}\right) \quad (1.15)$$

Taking the integral over the separatrix, the total stable phase-space area can be calculated ($\phi \cdot \delta$ units):

$$\mathcal{A}_0 = 2 \int_{-\pi}^{\pi} \frac{2}{h |\eta|} \frac{\omega_s}{\omega_{rev}} \cos\left(\frac{\phi}{2}\right) d\phi = \frac{16}{h |\eta|} \frac{\omega_s}{\omega_{rev}} \quad (1.16)$$

To obtain the phase-space area in eV·s, one should multiply this quantity by the reference momentum p_0 and divide by the RF frequency $2\pi f_{rf}$.

The region within the separatrix is known as the RF bucket. Particles within this region of phase-space stay within this region of phase-space. The average momentum and phase of particles in the RF bucket can be changed by adiabatically changing the fixed point

of the bucket. The collection of particles that share the same RF bucket are referred to as a bunch. A particle beam is composed of many discrete bunches, one arriving after another, all with similar momentum.

The size of a particle bunch in phase-space is described by quantity known as the longitudinal emittance, which calculated using the product of the RMS momentum spread with the RMS temporal spread:

$$\epsilon = \pi \sigma_p \sigma_T = \pi \frac{p_0}{2\pi f_{rf}} \sigma_\delta \sigma_\phi \quad (1.17)$$

A beam with a smaller longitudinal emittance is preferable because it facilitates the task of transporting the beam in the RF bucket. Nonlinear effects can distort or reduce the effective RF bucket area.

1.4 Synchrotron Oscillation Motion

In this section we examine the longitudinal dynamics of particles within the RF bucket. First we will show the shift in the synchrotron oscillation frequency with the synchrotron oscillation amplitude of a particle. Next we will show a perturbative solution to oscillatory particle trajectories.

The synchrotron oscillation period can be calculated as a function of maximum oscillation phase $\hat{\phi}$ by manipulating Eq. 1.11:

$$\begin{aligned} -\omega_s^2 \sin(\phi) &= \frac{d}{dt} \left[\frac{d\phi}{dt} \right] = \frac{d}{d\phi} \left[\frac{d\phi}{dt} \right] \frac{d\phi}{dt} = \frac{1}{2} \frac{d}{d\phi} \left[\left(\frac{d\phi}{dt} \right)^2 \right] \\ \frac{dt}{d\phi} &= \left[\int -2\omega_s^2 \sin(\phi) d\phi \right]^{-1/2} = \frac{1}{\sqrt{2}\omega_s} \frac{1}{\sqrt{\cos(\phi) - \cos(\hat{\phi})}} \\ T &= \frac{4}{\sqrt{2}\omega_s} \int_0^{\hat{\phi}} \frac{d\phi}{\sqrt{\cos(\phi) - \cos(\hat{\phi})}} = T_0 \frac{2}{\pi} K \left[\sin \left(\frac{\hat{\phi}}{2} \right) \right] \end{aligned} \quad (1.18)$$

where T_0 is the synchrotron period of small oscillations $T_0 = 2\pi/\omega_s$ and K is the complete elliptic integral of the first kind $K[k] = \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2(u)}}$.

To further understand how the large oscillation trajectory differs from the small oscillation trajectory, we can expand the system perturbatively. For small ϕ , Eq. 1.11 becomes:

$$\ddot{\phi} = -\omega_s^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \phi^{2k+1} \approx -\omega_s^2 \left(\phi - \frac{1}{6} \phi^3 + \dots \right) \quad (1.19)$$

Using the Poincare-Lindstedt method (see Ch. 2 of [3]), we can obtain a perturbative solution for Eq. 1.19 given by:

$$\phi \approx \rho \sin(\omega_s(1+\sigma)t + \psi) + A_3 \sin(3\omega_s(1+\sigma)t + 3\psi) + \dots \quad (1.20)$$

For brevity let $s_n \equiv \sin(n\omega_s(1+\sigma)t + n\psi)$. We put Eq. 1.20 into Eq. 1.19 to obtain:

$$\begin{aligned} \ddot{\phi} &\approx -\omega_s^2 \left[(\rho s_1 + A_3 s_3) - \frac{1}{6} (\rho^3 s_1^3) \right] \\ \ddot{\phi} &\approx -\omega_s^2 \left[(\rho s_1 + A_3 s_3) - \frac{1}{6} \left(\frac{3}{4} \rho^3 s_1 - \frac{1}{4} \rho^3 s_3 \right) \right] \end{aligned} \quad (1.21)$$

Applying the second-order derivation to Eq. 1.20 and setting the two sides equal, we obtain two equations:

$$-\omega_s^2(1+\sigma)^2 \rho s_1 = -\omega_s^2 \left(\rho - \frac{1}{8} \rho^3 \right) s_1 \quad (1.22)$$

$$-9\omega_s^2(1+\sigma)^2 A_3 s_3 = -\omega_s^2 \left(A_3 - \frac{1}{24} \rho^3 \right) s_3 \quad (1.23)$$

In Eq. 1.23 σ is negligible and we obtain:

$$A_3 = -\frac{1}{192} \rho^3 \quad (1.24)$$

We subtract $-\omega_s \rho s_1$ from each side of Eq. 1.22, hold σ^2 negligible, and solve for σ :

$$\sigma = -\frac{1}{16} \rho^2 \quad (1.25)$$

If we perform this expansion to higher orders of ρ we can obtain terms with higher odd harmonics of ω_s , obtain more precise calculations of the coefficients to these terms, and obtain a more precise calculation of the synchrotron tune shift σ .

1.5 Slipping Particles Motion

In this section we examine the longitudinal dynamics of particles outside of the RF bucket by calculating the perturbative solution to the particle trajectories.

Particles with trajectories outside of the separatrix slip with respect to the RF bucket. The motion of these particles can be studied by writing Eq. 1.11 in a moving reference frame:

$$\phi = \Omega t + \phi_o + \theta \quad (1.26)$$

$$\ddot{\theta} = -\omega_s^2 \sin(\Omega t + \phi_o + \theta) \quad (1.27)$$

$$\ddot{\theta} = -\omega_s^2 [\sin(\Omega t + \phi_o) \cos(\theta) + \cos(\Omega t + \phi_o) \sin(\theta)] \quad (1.28)$$

For small θ , we expand Eq. 1.26 perturbatively to study the oscillatory motion of the slipping particle:

$$\ddot{\theta} = -\omega_s^2 \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \theta^{2k} \sin(\Omega t + \phi_o) + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \theta^{2k+1} \cos(\Omega t + \phi_o) \right] \quad (1.29)$$

$$\ddot{\theta} \approx -\omega_s^2 [\sin(\Omega t + \phi_o) + \theta \cos(\Omega t + \phi_o)] \quad (1.30)$$

The perturbative solution to Eq. 1.29 is of the form:

$$\theta \approx B_1 \sin(\Omega t + \phi_o) + B_2 \sin(2\Omega t + 2\phi_o) + \dots \quad (1.31)$$

For brevity let $S_n \equiv \sin(n\Omega t + n\phi_o)$ and $C_n \equiv \cos(n\Omega t + n\phi_o)$. We put Eq. 1.31 into Eq. 1.29 to obtain:

$$\ddot{\theta} \approx -\omega_s^{-2} [S_1 + B_1 S_1 C_1] \quad (1.32)$$

We split Eq. 1.32 into two equations:

$$-\Omega^2 B_1 S_1 = -\omega_s^{-2} S_1 \quad (1.33)$$

$$-4\Omega^2 B_2 S_2 = -\frac{1}{2}\omega_s^{-2} B_1 S_2 \quad (1.34)$$

From Eq. 1.33 we obtain:

$$B_1 = \left(\frac{\omega_s}{\Omega}\right)^2 \quad (1.35)$$

From Eq. 1.34 and Eq. 1.35 we obtain:

$$B_2 = \frac{1}{8} \left(\frac{\omega_s}{\Omega}\right)^2 B_1 \approx \frac{1}{8} \left(\frac{\omega_s}{\Omega}\right)^4 \quad (1.36)$$

The perturbation can be expanded to higher orders and the coefficients of B_n are of the order $\left(\frac{\omega_s}{\Omega}\right)^{2n}$. The perturbation for small θ is better expressed as a perturbation of small $\left(\frac{\omega_s}{\Omega}\right)^2$. If the slipping particle trajectory is close to the separatrix then Ω is small and the nonlinear oscillatory motion is large. For a particle far from the reference momentum, the force from the RF cavity does not add up coherently and the motion approaches that of a coasting beam.

The motion of stable slip-stacking particles, which will be described in Chapter 3, is a combination of synchrotron motion and slipping motion.

1.6 Acceleration & Focusing Motion

Particle beams are accelerated by gradually changing the resonant frequency of the RF cavities. The momentum of the stable particles change to match the new reference momentum. For a linearly increasing RF frequency, we transform the coordinates into the accelerating reference frame:

$$\dot{\delta} = f_{rev} V_{\delta} [\sin(\phi) - \sin(\phi_s)], \quad \dot{\phi} = 2\pi f_{rev} h \eta \delta \quad (1.37)$$

The stable fixed point for these equations of motions is $\phi = \phi_s, \delta = 0$, which corresponds to a particle whose momentum exactly follows the reference momentum and whose phase changes each revolution to exactly match the change in the RF frequency. The separatrix changes depending on ϕ_s and this region of phase-space is referred to as the running RF

bucket. The greater the magnitude of ϕ_s , the greater the acceleration and the smaller the RF bucket.

Fig. 1.5 shows the motion of particles inside and outside the running RF bucket. Particles within the separatrix remain synchronized with the RF and accelerate linearly. Particles outside the separatrix, however, are not accelerated and deviate increasingly from the accelerating reference momentum. The transversely bending and focusing magnetic elements of the particle accelerator ring increase in field strength to follow the accelerating beam and consequently the particles that are not accelerated are lost. The total range of momentum that a particle accelerator ring can support due to transverse dynamics is known as the momentum aperture of that ring. The total range of momentum that an RF bucket can store simultaneously is known as the momentum acceptance.

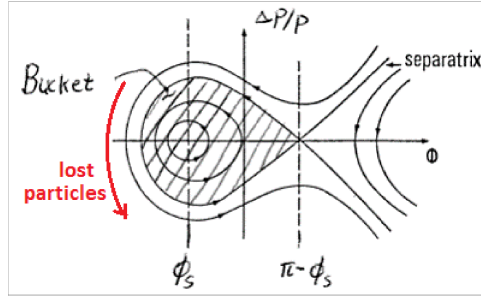


Figure 1.5: Phase space trajectories associated with the running bucket. Only particles inside the separatrix follow the accelerating reference frame.

In [2], the running bucket area factor $\alpha_b(\phi_s)$ gives the ratio between the stable phase-space area of the running bucket and the stationary bucket:

$$\alpha_b(\phi_s) = \frac{\mathcal{A}_b}{\mathcal{A}_0} = \frac{1}{4\sqrt{2}} \int_{\phi_u}^{\pi-\phi_s} |\cos(\phi) + \cos(\phi_s) - (\pi - \phi - \phi_s) \sin(\phi_s)|^{1/2} d\phi \quad (1.38)$$

where ϕ_u is found from the transcendental equation

$$\cos(\phi_u) + \phi_u \sin(\phi_s) = -\cos(\phi_s) + (\pi - \phi_s) \sin(\phi_s) \quad (1.39)$$

The running bucket area factor is plotted in Fig. 1.6 and is approximated with

$$\alpha_b(\phi_s) \approx \frac{1 - \sin(\phi_s)}{1 + \sin(\phi_s)} \quad (1.40)$$

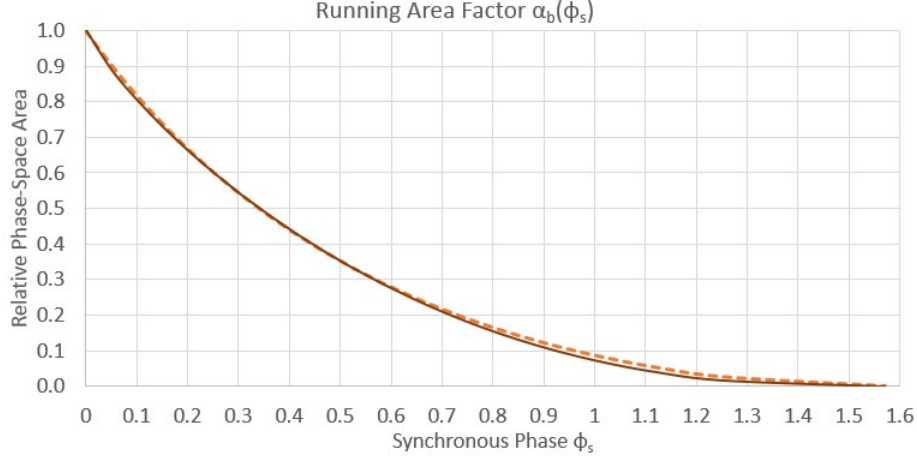


Figure 1.6: The ratio between the running bucket area and the corresponding stationary bucket area, as a function of the synchronous phase ϕ_s of the running bucket. The exact value is shown in a dark orange line and the approximate value is shown in a dashed orange.

The parameters f_{rev} , V_δ , and η in Eq. 1.37 generally change with acceleration but can be treated adiabatically so long as they change slowly compared to the synchrotron frequency. This approximation breaks down when η is small because the focusing is weaker and this is referred to as quasi-isosynchronous condition. In the quasi-isosynchronous case, the dependency of the revolution frequency on η is expanded to include the second order term:

$$\frac{T - T_{rev}}{T_{rev}} \approx 0 + \frac{1}{T_{rev}} \frac{\partial T}{\partial \delta} \delta + \frac{1}{T_{rev}} \frac{\partial^2 T}{\partial \delta^2} \frac{\delta^2}{2} = \eta_0 \delta + \eta_1 \delta^2 \quad (1.41)$$

When a particle accelerator crosses transition energy, where $\eta = 0$, it can be a significant source of longitudinal emittance growth. At transition energy, acceleration is still occurs but there is no linear longitudinal focusing force. After crossing transition energy, the longitudinal focusing force will have changed orientation and the stable fixed point will have

changed by π . The process complicates particle motion and decoherence in the ensemble of particles increases the longitudinal emittance.

1.7 Summary

In this chapter we show that longitudinal dynamics of single fixed-frequency RF cavity is identical to that of simple pendulum. The separatrix of the simple pendulum separates the stable phase-space area from the unstable phase-space area. The stable particles are in the RF bucket and the unstable particles are slipping with respect to the RF bucket. The ensemble of particles that share the same RF bucket are referred to as a bunch and the size of a bunch is measured by its longitudinal emittance. In order to change the energy of a particle bunch the RF frequency must be changed gradually and only the particles inside the separatrix will be accelerated successfully.